

# Completely regular codes with different parameters and the same distance-regular coset graphs

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## Abstract

A known Kronecker construction of completely regular codes has been investigated taking different alphabets in the component codes. This approach is also connected with lifting constructions of completely regular codes. We obtain several classes of completely regular codes with different parameters, but identical intersection array. Given a prime power  $q$  and any two natural numbers  $a, b$ , we construct completely transitive codes over different fields with covering radius  $\rho = \min\{a, b\}$  and identical intersection array, specifically, one code over  $\mathbb{F}_{q^r}$  for each divisor  $r$  of  $a$  or  $b$ . As a corollary, for any prime power  $q$ , we show that distance regular bilinear forms graphs can be obtained as coset graphs from several completely regular codes with different parameters. Under the same conditions, an explicit construction of an infinite family of  $q$ -ary uniformly packed codes (in the wide sense) with covering radius  $\rho$ , which are not completely regular, is also given.

*Keywords:* bilinear forms graph, completely regular code, completely transitive code, coset graph, distance-regular graph, distance-transitive graph, Kronecker product construction, lifting of a field, uniformly packed code

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## 1. Introduction

Let  $\mathbb{F}_q$  be a finite field of the order  $q$  and  $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$ . A  $q$ -ary linear code  $C$  of length  $n$  is a  $k$ -dimensional subspace of  $\mathbb{F}_q^n$ . Given any vector  $\mathbf{v} \in \mathbb{F}_q^n$ , its *distance to the code  $C$*  is  $d(\mathbf{v}, C) = \min_{\mathbf{x} \in C} \{d(\mathbf{v}, \mathbf{x})\}$ , the *minimum distance* of the code is  $d = \min_{\mathbf{v} \in C} \{d(\mathbf{v}, C \setminus \{\mathbf{v}\})\}$  and the *covering radius* of the code  $C$  is  $\rho = \max_{\mathbf{v} \in \mathbb{F}_q^n} \{d(\mathbf{v}, C)\}$ . We say that  $C$  is a  $[n, k, d; \rho]_q$ -code. Let  $D = C + \mathbf{x}$  be a *coset* of  $C$ , where  $+$  means the component-wise addition in  $\mathbb{F}_q$ . The *weight*  $\text{wt}(D)$  of  $D$  is the minimum weight of the codewords of  $D$ .

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For a given  $q$ -ary code  $C$  with covering radius  $\rho = \rho(C)$  define

$$C(i) = \{\mathbf{x} \in \mathbb{F}_q^n : d(\mathbf{x}, C) = i\}, \quad i = 1, 2, \dots, \rho.$$

Say that two vectors  $\mathbf{x}$  and  $\mathbf{y}$  are *neighbors* if  $d(\mathbf{x}, \mathbf{y}) = 1$ . For two vectors  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_n)$  over  $\mathbb{F}_q$  denote by  $\mathbf{xy}$  their inner product over  $\mathbb{F}_q$ , i.e.

$$\mathbf{xy} = x_1y_1 + \dots + x_ny_n.$$

**Definition 1.1.** [10] A  $q$ -ary code  $C$  is *completely regular*, if for all  $l \geq 0$  every vector  $x \in C(l)$  has the same number  $c_l$  of neighbors in  $C(l-1)$  and the same number  $b_l$  of neighbors in  $C(l+1)$ . Define  $a_l = (q-1)n - b_l - c_l$  and  $c_0 = b_\rho = 0$ . Denote by  $(b_0, \dots, b_{\rho-1}; c_1, \dots, c_\rho)$  the *intersection array* of  $C$ .

The equivalent definition of completely regular codes is due to Delsarte [4].

**Definition 1.2.** [4] A  $q$ -ary code  $C$  with covering radius  $\rho$  is called *completely regular* if the weight distribution of any translated  $D$  of  $C$  of weight  $i$ ,  $i = 0, 1, \dots, \rho$  is uniquely defined by the minimum weight of  $D$ , i.e. by the number  $i = \text{wt}(D)$ .

Let  $M$  be a monomial matrix, i.e. a matrix with exactly one nonzero entry in each row and column. If  $q$  is prime, then  $\text{Aut}(C)$  consist of all monomial  $(n \times n)$ -matrices  $M$  over  $\mathbb{F}_q$  such that  $\mathbf{c}M \in C$  for all  $\mathbf{c} \in C$ . If  $q$  is a power of a prime number, then  $\text{Aut}(C)$  also contains any field automorphism of  $\mathbb{F}_q$  which preserves  $C$ . The group  $\text{Aut}(C)$  acts on the set of cosets of  $C$  in the following way: for all  $\sigma \in \text{Aut}(C)$  and for every vector  $\mathbf{v} \in \mathbb{F}_q^n$  we have  $(\mathbf{v} + C)^\sigma = \mathbf{v}^\sigma + C$ .

**Definition 1.3.** [7, 14] Let  $C$  be a linear code over  $\mathbb{F}_q$  with covering radius  $\rho$ . Then  $C$  is *completely transitive* if  $\text{Aut}(C)$  has  $\rho + 1$  orbits when acts on the cosets of  $C$ .

Since two cosets in the same orbit should have the same weight distribution, it is clear, that any completely transitive code is completely regular.

**Definition 1.4.** [1] Let  $C$  be a  $q$ -ary code of length  $n$  and let  $\rho$  be its covering radius. We say that  $C$  is *uniformly packed in the wide sense*, i.e. in the sense of [1], if there exist rational numbers  $\alpha_0, \dots, \alpha_\rho$  such that for any  $\mathbf{v} \in \mathbb{F}_q^n$

$$\sum_{k=0}^{\rho} \alpha_k f_k(\mathbf{v}) = 1, \quad (1)$$

where  $f_k(\mathbf{v})$  is the number of codewords at distance  $k$  from  $\mathbf{v}$ .

Completely regular and completely transitive codes are classical subjects in algebraic coding theory, which are closely connected with graph theory, combinatorial designs and algebraic combinatorics. Existence, construction and

enumeration of all such codes are open hard problems (see [3, 10, 6, 8] and references there).

In a recent paper [12] we described an explicit construction, based on the Kronecker product of parity check matrices, which provides, for any natural number  $\rho$  and for any prime power  $q$ , an infinite family of  $q$ -ary linear completely regular codes with covering radius  $\rho$ . In [13] we presented another class of  $q$ -ary linear completely regular codes with the same property, based on lifting of perfect codes. Here we extend the Kronecker product construction to the case when component codes have different alphabets and connect the resulting completely regular codes with codes obtained by lifting  $q$ -ary perfect codes. This gives several different infinite classes of completely regular codes with different parameters and with identical intersection arrays.

## 2. Preliminary results

For a  $q$ -ary  $[n, k, d; \rho]_q$  code  $C$  let  $s = s(C)$  be its *outer distance*, i.e. the number of different nonzero weights of codewords in the dual code  $C^\perp$ .

**Lemma 2.1.** *Let  $C$  be a code with covering radius  $\rho$  and external distance  $s$ . Then,*

- i) [4]  $\rho(C) \leq s(C)$ .
- ii) [2] *The code  $C$  is uniformly packed in the wide sense if, and only if,  $\rho = s$ .*
- iii) [3] *If  $C$  is completely regular then it is uniformly packed in the wide sense.*

**Lemma 2.2.** [10] *Let  $C$  be a linear completely regular code with intersection array  $(b_0, \dots, b_{\rho-1}; c_1, \dots, c_\rho)$ , and let  $\mu_i$  be the number of cosets of  $C$  of weight  $i$ . Then*

$$\mu_i b_i = \mu_{i+1} c_{i+1}.$$

**Definition 2.3.** *For two matrices  $A = [a_{r,s}]$  and  $B = [b_{i,j}]$  over  $\mathbb{F}_q$  define a new matrix  $H$  which is the Kronecker product  $H = A \otimes B$ , where  $H$  is obtained by changing any element  $a_{r,s}$  in  $A$  by the matrix  $a_{r,s}B$ .*

Consider the matrix  $H = A \otimes B$  and let  $C$ ,  $C_A$  and  $C_B$  be the codes over  $\mathbb{F}_q$  which have, respectively,  $H$ ,  $A$  and  $B$  as parity check matrices. Assume that  $A$  and  $B$  have size  $m_a \times n_a$  and  $m_b \times n_b$ , respectively. For  $r \in \{1, \dots, m_a\}$  and  $s \in \{1, \dots, m_b\}$  the rows in  $H$  look as

$$(a_{r,1}b_{s,1}, \dots, a_{r,1}b_{s,n_b}, a_{r,2}b_{s,1}, \dots, a_{r,2}b_{s,n_b}, \dots, a_{r,n_a}b_{s,1}, \dots, a_{r,n_a}b_{s,n_b}).$$

Arrange these rows taking blocks of  $n_b$  coordinates as columns such that the codewords in code  $C$  are presented as matrices  $[\mathbf{c}]$  of size  $n_b \times n_a$ :

$$[\mathbf{c}] = \begin{bmatrix} c_{1,1} & \dots & c_{1,n_a} \\ c_{2,1} & \dots & c_{2,n_a} \\ \vdots & \vdots & \vdots \\ c_{n_b,1} & \dots & c_{n_b,n_a} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_{n_b} \end{bmatrix} = [\mathbf{c}^{(1)} \mathbf{c}^{(2)} \dots \mathbf{c}^{(n_a)}], \quad (2)$$

where  $c_{i,j} = a_{r,j}b_{s,i}$ ,  $\mathbf{c}_r$  is the  $r$ th row vector of the matrix  $C$  and  $\mathbf{c}^{(\ell)}$  is its  $\ell$ th column.

The following result was obtained in [12].

**Theorem 2.4.** *Let  $C(H)$  be the  $[n, k, d; \rho]_q$  code with parity check matrix  $H = A \otimes B$  where  $A$  and  $B$  are parity check matrices of Hamming  $[n_a, k_a, 3]_q$  and  $[n_b, k_b, 3]_q$  codes,  $C_A$  and  $C_B$ , respectively, where  $n_a = (q^{m_a} - 1)/(q - 1) \geq 3$ ,  $n_b = (q^{m_b} - 1)/(q - 1) \geq 3$ ,  $k_a = n_a - m_a$ ,  $k_b = n_b - m_b$  and where*

$$n = n_a n_b, \quad k = n - m_a m_b, \quad d = 3, \quad \rho = \min\{m_a, m_b\}.$$

*Then the code  $C$  is completely transitive and, therefore, completely regular with covering radius  $\rho = \min\{m_a, m_b\}$  and intersection numbers*

$$\begin{aligned} b_\ell &= \frac{(q^{m_a} - q^\ell)(q^{m_b} - q^\ell)}{(q - 1)}, \quad \ell = 0, \dots, \rho - 1, \\ c_\ell &= q^{\ell-1} \frac{q^\ell - 1}{q - 1}, \quad \ell = 1, \dots, \rho. \end{aligned} \tag{3}$$

**Definition 2.5.** *Let  $C$  be the  $[n, k, d]_q$  code with parity check matrix  $H$  where  $1 \leq k \leq n - 1$  and  $d \geq 3$ . Denote by  $C_r$  the  $[n, k, d]_{q^r}$  code over  $\mathbb{F}_{q^r}$  with the same parity check matrix  $H$ . Say that code  $C_r$  is obtained by lifting  $C$  to  $\mathbb{F}_{q^r}$ .*

In [13] we proved the following result

**Theorem 2.6.** *Let  $C_r(H_m^q)$  be the  $[n, n - m, 3; \rho]_{q^r}$  code of length  $n = (q^m - 1)/(q - 1)$  over the field  $\mathbb{F}_{q^r}$  obtained by lifting a  $q$ -ary perfect  $[n, n - m, 3]_q$  code  $C(H_m^q)$  with parity check matrix  $H_m^q$ . Then, code  $C_r(H_m^q)$  is completely regular with covering radius  $\rho = \min\{m, r\}$  and intersection numbers given by (3) taking  $m_a = m$  and  $m_b = r$ .*

From the above theorems we are giving completely regular codes with different parameters and over different alphabets but with the same intersection arrays. Now our purpose is to consider deeper this coincidence.

Let  $H_m^q = [\mathbf{h}^{(1)} | \dots | \mathbf{h}^{(n)}]$  be a parity check matrix for the  $q$ -ary perfect  $[n, n - m, 3]_q$  code  $C = C(H_m^q)$ . Any codeword  $\mathbf{v}$  of  $C$  is defined by the following equation

$$v_1 \mathbf{h}^{(1)} + v_2 \mathbf{h}^{(2)} + \dots + v_n \mathbf{h}^{(n)} = 0, \quad v_i \in \mathbb{F}_q. \tag{4}$$

A codeword  $\mathbf{v}$  of the lifted code  $C_r(H_m^q)$  is defined by the same equation

$$v_1 \mathbf{h}^{(1)} + v_2 \mathbf{h}^{(2)} + \dots + v_n \mathbf{h}^{(n)} = 0, \tag{5}$$

with the only difference that the unknown elements  $v_i$  belong to  $\mathbb{F}_{q^r}$ . Since any element  $v_i$  of  $\mathbb{F}_{q^r}$  can be presented as a vector  $\mathbf{v}_i = (v_{i,1}, v_{i,2}, \dots, v_{i,r})$  of length  $r$  over  $\mathbb{F}_q$ , the equation (5) is transformed to the system of equations

$$v_{1,\ell} \mathbf{h}^{(1)} + v_{2,\ell} \mathbf{h}^{(2)} + \dots + v_{n,\ell} \mathbf{h}^{(n)} = 0, \quad \ell = 1, \dots, r. \tag{6}$$

Since for any  $\ell$ ,  $\ell \in \{1, \dots, r\}$ , the solutions  $(v_{1,\ell}, v_{2,\ell}, \dots, v_{n,\ell})$  of (6) are also solutions of (4), we conclude that (5) has  $(q^{n-m})^r = q^{(n-m)r}$  solutions. Indeed, matrix  $H_m^q$  has  $n - m$  linearly independent rows.

Now, consider a code  $D$  obtained by the Kronecker product, i.e.  $D$  has a parity check matrix  $H = A \otimes B$ . So, column vectors of  $H$  can be seen as  $\mathbf{a}^{(i)} \otimes \mathbf{b}^{(j)}$ . Assume that  $A$  is of size  $m \times n$  and  $B$  is of size  $m_b \times n_b$ , where both matrices are over  $\mathbb{F}_q$ . So, a codeword  $\mathbf{v}$  of  $D$

$$\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n), \quad \mathbf{v}_i = (v_{i,1}, \dots, v_{i,n_b}),$$

is defined by the equation

$$\sum_{i=1}^n \mathbf{a}^{(i)} \sum_{j=1}^{n_b} v_{i,j} \mathbf{b}^{(j)} = 0, \quad (7)$$

or by the system of equations

$$\sum_{i=1}^n \mathbf{a}^{(i)} \sum_{j=1}^{n_b} v_{i,j} b_s^{(j)} = 0, \quad s = 1, \dots, m_b, \quad (8)$$

which can be rewritten as follows:

$$w_{1,s} \mathbf{a}^{(1)} + w_{2,s} \mathbf{a}^{(2)} + \dots + w_{n,s} \mathbf{a}^{(n)} = 0, \quad s = 1, \dots, m_b, \quad (9)$$

where

$$w_{i,s} = \sum_{j=1}^{n_b} v_{i,j} b_s^{(j)}. \quad (10)$$

Now the following observation gives an explanation of the coincidence of the intersection numbers of both constructions: *when vector  $(v_{i,1}, v_{i,2}, \dots, v_{i,n_b})$  in (10) runs over all possible values in  $\mathbb{F}_q^{n_b}$ , vector  $(w_{i,1}, w_{i,2}, \dots, w_{i,m_b})$  runs over all possible values in  $\mathbb{F}_q^{m_b}$ .* It is so, because the matrix  $B$  is of size  $m_b \times n_b$  and has rank  $m_b$ . Hence the linear system (10) defines an homomorphism of vectors from  $\mathbb{F}_q^{n_b}$  onto  $\mathbb{F}_q^{m_b}$ , whose kernel has cardinality  $q^{n_b - m_b}$ .

Compare the equations (6) and (9). If  $r = m_b$  they look identically. So, for the graphs defined by the coset graphs we have the same conditions. But the corresponding codes are different (since they have different lengths). The difference is that in (9) the variables  $w_{i,s}$  are not codewords of  $D$ , but define the codewords through the linear system (10). Hence, from the point of view of the shape of parity check equations (in both cases, the columns,  $\mathbf{h}^{(i)}$  and  $\mathbf{a}^{(i)}$ , are all linearly independent vectors of corresponding lengths), the Kronecker product construction can be considered as a special type of the lifting construction. This is an explanation why the intersection arrays of different completely regular codes from different constructions coincide (compare Theorems 2.4 and 2.6).

### 3. Extending the Kronecker product construction

Recall that by  $C(H)$  we denote the code defined by the parity check matrix  $H$ , by  $H_m^q$  we denote the parity check matrix of the  $q$ -ary Hamming  $[n, n-m, 3]_q$  code  $C = C(H_m^q)$  of length  $n = (q^m - 1)/(q - 1)$ , and by  $C_r(H_m^q)$  we denote the code (of the same length  $n = (q^m - 1)/(q - 1)$ ) obtained by lifting  $C(H_m^q)$  to the field  $\mathbb{F}_{q^r}$ .

Considering the above Kronecker construction (Theorem 2.4) we could see that the alphabets of both matrices  $A = [a_{i,j}]$  and  $B$  should be compatible to each other in the sense that the multiplication  $a_{i,j}B$  can be carried out. To have this compatibility it is enough that, say, the matrix  $A$  is over  $\mathbb{F}_{q^u}$  and  $B$  is over  $\mathbb{F}_q$ . First, we consider the covering radius of the resulting codes.

**Lemma 3.1.** *Let  $C(H_{m_a}^{q^u})$  and  $C(H_{m_b}^q)$  be two Hamming codes with parameters  $[n_a, n_a - m_a, 3]_{q^u}$  and  $[n_b, n_b - m_b, 3]_q$ , respectively, where  $n_a = (q^{um_a} - 1)/(q^u - 1)$ ,  $n_b = (q^{m_b} - 1)/(q - 1)$ ,  $q$  is a prime power,  $m_a, m_b \geq 2$ , and  $u \geq 1$ . Then the code  $C$  with parity check matrix  $H = H_{m_a}^{q^u} \otimes H_{m_b}^q$ , the Kronecker product of  $H_{m_a}^{q^u}$  and  $H_{m_b}^q$ , has covering radius  $\rho = \min\{u m_a, m_b\}$ .*

PROOF. Assume that the matrices  $H$ ,  $H_{m_a}^{q^u}$ , and  $H_{m_b}^q$  have columns  $\mathbf{h}_i$ ,  $\mathbf{a}_j$ , and  $\mathbf{b}_s$ , respectively, i.e. they look as

$$H = [\mathbf{h}_1 | \dots | \mathbf{h}_n], \quad H_{m_a}^{q^u} = [\mathbf{a}_1 | \dots | \mathbf{a}_{n_a}], \quad H_{m_b}^q = [\mathbf{b}_1 | \dots | \mathbf{b}_{n_b}].$$

We have to prove that any column vector  $\mathbf{x} \in (\mathbb{F}_{q^u})^{m_a m_b}$  can be presented as a linear combination of not more than  $\rho$  columns of  $H$ .

By construction the column  $\mathbf{h}_i$  looks as

$$\mathbf{h}_i^T = [a_{1,j} \mathbf{b}_s, a_{2,j} \mathbf{b}_s, \dots, a_{m_a,j} \mathbf{b}_s],$$

where  $i = 1, \dots, n$ ,  $n = n_a n_b$ ,  $j = 1, \dots, n_a$ ,  $s = 1, \dots, n_b$  and  $\mathbf{h}_i^T$  is the transposed vector of  $\mathbf{h}_i$ . By definition, the matrix  $H_{m_b}^q$  contains as column vectors any vector  $\mathbf{y} \in (\mathbb{F}_q)^{m_b}$  over the ground field  $\mathbb{F}_q$ , up to multiplication by scalars of  $\mathbb{F}_q$ . But, vectors  $\mathbf{x}$  are arbitrary vectors over the extended field  $\mathbb{F}_{q^u}$ . Any such vector can be presented as a linear combination of  $u$  or less vectors from  $H_{m_b}^q$ . Hence for any choice of  $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_{m_a})$  we can always take not more than  $u m_a$  columns of  $H$  to have  $m_a$  equalities of the type

$$\mathbf{x}_i = \sum_{s=1}^u \alpha_s \mathbf{b}_{i_s}, \quad i = 1, \dots, m_a, \quad \alpha_i \in \mathbb{F}_{q^u},$$

implying that

$$\rho \leq u m_a.$$

From the other side, by permuting the rows of  $H$ , the column  $\mathbf{h}_i$  can be presented (with other index, say,  $i'$ ) as follows:

$$\mathbf{h}_{i'}^T = [b_{1,s} \mathbf{a}_j, b_{2,s} \mathbf{a}_j, \dots, b_{m_b,s} \mathbf{a}_j].$$

Since vector  $\mathbf{a}_j$  is over the extended field  $\mathbb{F}_{q^u}$ , we can choose as  $\mathbf{y} \in (\mathbb{F}_{q^u})^{m_a}$  (a component of  $\mathbf{x}$ ) a vector  $\mathbb{F}_{q^u}$ , which can be presented only as some vectors  $\mathbf{a}_j$ , up to scalar, giving that

$$\rho \leq m_b.$$

Since in both cases the bounds can be reached by appropriate choices of vector  $\mathbf{x}$ , we obtain the result.  $\square$

We give also several simple facts from [12, 13], which will be used in the proof of the forthcoming theorem. As we said before (2), any codeword  $\mathbf{c} \in C$  can be seen as a  $(n_b \times n_a)$ -matrix  $[\mathbf{c}]$ .

For a codeword  $[\mathbf{c}]$  define its syndrome, which, in a matrix representation, is a  $(m_b \times m_a)$  matrix  $S_c = [(A \otimes B)\mathbf{c}^T]$  which is equal to zero. We have

$$S_c = [(A \otimes B)\mathbf{c}^T] = B[\mathbf{c}]A = 0.$$

From the equality above we see that any  $(n_b \times n_a)$ -matrix having codewords of  $C(A)$  as rows belongs to the code  $C$ , and any  $(n_b \times n_a)$ -matrix with codewords of  $C(B)$  as columns also belongs to the code  $C$ . And vice versa, all the codewords in  $C$  can always be seen as linear combinations of matrices of the both types above.

Fix a 1 – 1 mapping  $\mu$  from  $\mathbb{F}_{q^u}$  to  $(\mathbb{F}_q)^u$  writing for any element  $a \in \mathbb{F}_{q^u}$  its  $\mathbb{F}_q$ -presentation  $\mu(a)$ :

$$\mu(a) = [a_0, a_1, \dots, a_{u-1}] \longleftrightarrow a = \sum_{i=0}^{u-1} a_i \mu_i,$$

where  $\mu_0, \mu_1, \dots, \mu_{u-1}$  is a fixed basis in  $\mathbb{F}_{q^u}$  over  $\mathbb{F}_q$ . Finally, extend the map  $\mu$  to vectors  $\mathbf{v} = (v_1, \dots, v_n) \in (\mathbb{F}_{q^u})^n$  by the obvious way:  $\mu(\mathbf{v}) = [\mu(v_1) \mid \dots \mid \mu(v_n)]$ .

**Definition 3.2.** Given a vector  $\mathbf{v} \in (\mathbb{F}_{q^u})^n$ , with syndrome  $S_v$ , which is a  $(m_b \times m_a)$  matrix over  $\mathbb{F}_{q^u}$  denote by  $\mu_v$  the  $(m_b \times (um_a))$  matrix obtained from  $S_v$  using the map  $\mu$  in its rows.

We have the following three simple facts, which we formulate into the next lemmas.

**Lemma 3.3.** Let  $\mathbf{x}, \mathbf{y} \in (\mathbb{F}_{q^u})^n$  be two vectors with syndromes  $S_x$  and  $S_y$  and corresponding matrices  $\mu_x$  and  $\mu_y$ , respectively. The equality

$$(\mu_x)^T K = \mu_y,$$

for any non-singular  $m_b \times m_b$  matrix over  $\mathbb{F}_q$  implies the equality

$$(S_x)^T K = S_y$$

and vice versa.

**Lemma 3.4.** *Let  $H$  be any  $(m \times n)$  matrix over  $\mathbb{F}_q$  and  $C_q(H)$  (respectively,  $C_{q^u}(H)$ ) be the code over  $\mathbb{F}_q$  (respectively, over  $\mathbb{F}_{q^u}$ ) with parity check matrix  $H$ . Then:*

- i) *Any vector  $\mathbf{v} = (v_1, \dots, v_n) \in (\mathbb{F}_{q^u})^n$  is a codeword of  $C_{q^u}(H)$  if and only if all rows of the matrix  $[\mu(v_1)^T \mid \dots \mid \mu(v_n)^T]$  are codewords of  $C_q(H)$ .*
- ii) *If  $\varphi$  is an automorphism of  $C_q(H)$ , then it is an automorphism of  $C_{q^u}(H)$ .*

The following statement generalizes the results of [12, 13].

**Theorem 3.5.** *Let  $C(H_{m_a}^{q^u})$  and  $C(H_{m_b}^q)$  be two Hamming codes with parameters  $[n_a, n_a - m_a, 3]_{q^u}$  and  $[n_b, n_b - m_b, 3]_q$ , respectively, where  $n_a = (q^{u m_a} - 1)/(q^u - 1)$ ,  $n_b = (q^{m_b} - 1)/(q - 1)$ ,  $q$  is a prime power,  $m_a, m_b \geq 2$ , and  $u \geq 1$ .*

- i) *The code  $C$  with parity check matrix  $H = H_{m_a}^{q^u} \otimes H_{m_b}^q$ , the Kronecker product of  $H_{m_a}^{q^u}$  and  $H_{m_b}^q$ , is a completely transitive, and so completely regular,  $[n, k, d; \rho]_{q^u}$  code with parameters*

$$n = n_a n_b, \quad k = n - m_a m_b, \quad d = 3, \quad \rho = \min\{u m_a, m_b\}. \quad (11)$$

- ii) *The code  $C$  has the intersection numbers:*

$$b_\ell = \frac{(q^{u m_a} - q^\ell)(q^{m_b} - q^\ell)}{(q - 1)}, \quad \ell = 0, 1, \dots, \rho - 1,$$

and

$$c_\ell = q^{\ell-1} \frac{q^\ell - 1}{q - 1}, \quad \ell = 1, 2, \dots, \rho.$$

- iii) *The lifted code  $C_{m_b}(H_{u m_a}^q)$  is a completely regular code with the same intersection array as  $C$ .*

PROOF. The proof is mostly based on the same arguments which we used in the two previous papers [12, 13]. We shortly repeat only the places which it differ from the quoted papers.

First, from Lemma 3.1, we have that  $\rho = \min\{m_b, u m_a\}$ .

The next step is to prove that the code  $C$  is completely transitive. This part is coming from similar arguments, which we have used in the proof of [12, Theo. 1]. The only difference is that we have to use here Lemma 3.3 in order to guaranty the existence of the invertible  $m_b \times m_b$  matrix  $K$  over  $\mathbb{F}_q$  such that the equality  $S_x^T K = S_y^T$  holds where  $S_x$  and  $S_y$  are the syndromes of vectors  $\mathbf{x}$  and  $\mathbf{y}$  over  $\mathbb{F}_{q^u}$ .

Denote by  $C_A$  and  $C_B$  the codes over  $\mathbb{F}_{q^u}$ , with parity check matrices  $A$  and  $B$ , respectively.

To prove that  $C$  is a completely transitive code it is enough to show that starting from two vectors  $\mathbf{x}, \mathbf{y} \in C(\ell)$ ,  $1 \leq \ell \leq \rho$ , there exists a monomial matrix  $\varphi \in \text{Aut}(C)$  such that  $\mathbf{x}\varphi \in \mathbf{y} + C$  or, computing the syndrome (Lemma 3.3),

$$S_{\mathbf{x}\varphi} = [(A \otimes B)(\mathbf{x}\varphi)^T] = [(A \otimes B)(\mathbf{y})^T].$$



Let  $\phi_1$  be any monomial  $(n_a \times n_a)$  matrix and  $\phi_2$  be any monomial  $(n_b \times n_b)$  matrix. It is well known [9] that

$$(A\phi_1) \otimes (B\phi_2) = (A \otimes B)(\phi_1 \otimes \phi_2)$$

and  $\phi_1 \otimes \phi_2$  is a monomial  $(n_a n_b \times n_a n_b)$  matrix.

Note that if  $\varphi \in \text{Aut}(C)$  then  $H\varphi^T$  is a parity check matrix for  $C$  when  $H$  is. Therefore, taking the specific case where  $\phi_1^T \in \text{Aut}(C_A)$  and  $\phi_2^T \in \text{Aut}(C_B)$  we conclude that  $(\phi_1^T \otimes \phi_2^T)^T \in \text{Aut}(C)$ , or the same  $\phi_1 \otimes \phi_2 \in \text{Aut}(C)$ .

The two given vectors  $\mathbf{x}, \mathbf{y}$  belong to  $C(\ell)$  and so, from [12],  $\text{rank}(S_x) = \text{rank}(S_y) = \ell$ , where  $S_x$  and  $S_y$  are the syndrome of  $\mathbf{x}$  and  $\mathbf{y}$ , respectively. To prove that  $C$  is a completely transitive code we show that there exists a monomial matrix  $\phi^T \in \text{Aut}(C_B)$  such that

$$\begin{aligned} (A \otimes B)\mathbf{y}^T &= (A \otimes B\phi)\mathbf{x}^T \\ &= (A \otimes B)((I_{n_a} \otimes \phi)\mathbf{x}^T) \end{aligned}$$

where  $I_{n_a}$  is the  $n_a \times n_a$  identity matrix.

Since  $\ell \leq \rho \leq m_b$ , it is straightforward to find an invertible  $(m_b \times m_b)$  matrix  $K$  over  $\mathbb{F}_q$  such that  $\mu_x^T K = \mu_y^T$ . By Lemma 3.3 we conclude that  $S_x^T K = S_y^T$ . Since  $B$  is the parity check matrix of a Hamming code, the matrix  $K^T B$  is again a parity check matrix for a Hamming code and  $K^T B = B\phi$  for some monomial matrix  $\phi$ . Moreover, if  $G_B$  is the corresponding generator matrix for this Hamming code, i.e.  $B G_B^T = 0$ , then  $(B\phi)G_B^T = (K^T B)G_B^T = 0$  and so  $\phi^T \in \text{Aut}(C_B)$ .

Finally, we have

$$\begin{aligned} (A \otimes B)\mathbf{y}^T &= S_y = K^T S_x = K^T (B[\mathbf{x}]A^T) \\ &= B\phi[\mathbf{x}]A^T = (A \otimes B\phi)\mathbf{x}^T \\ &= (A \otimes B)((I_{n_a} \otimes \phi)\mathbf{x}^t). \end{aligned}$$

Since the code  $C$  is completely transitive we conclude that  $C$  is completely regular with the parameters (11). This gives item i). Now we have to write down the expressions for all intersection numbers. In this case we use the same approach as in [13].

We begin computing  $b_0$ , so the number of vectors in  $C(1)$  which are at distance one from one given vector in  $C$ . Without loss of generality (since  $C$  is a linear code we can fix the zero codeword  $\mathbf{0}$  in  $C$  and count how many different vectors of weight one there are in  $C(1)$ ). The answer is immediately

$$b_0 = n(q^u - 1) = \frac{(q^{u m_a} - 1)(q^{m_b} - 1)}{(q - 1)}.$$

Since the code  $C$  has minimum distance  $d = 3$ , we have  $c_1 = 1$ .

In general, let  $1 \leq i \leq \rho - 1$ . Take a  $(u m_a \times m_b)$ -matrix  $E$  of rank  $i$ , over  $\mathbb{F}_q$ , and compute the value  $b_i$  as the number of different  $(u m_a \times m_b)$ -matrices

$\bar{E}$ , over  $F_q$ , of rank  $i + 1 \leq \rho$ , such that  $E - \bar{E}$  has only one nonzero row. This value is well known (see reference in [13]):

$$b_i = \frac{(q^{u m_a} - q^i)(q^{m_b} - q^i)}{(q - 1)}.$$

Now, using the expressions for  $b_{i-1}$ ,  $\mu_i$  and  $\mu_{i-1}$  from Lemma 2.2, we obtain

$$c_i = \frac{\mu_{i-1} b_{i-1}}{\mu_i} = q^{i-1} \frac{(q^i - 1)}{q - 1},$$

i.e., we have item *ii*).

The last statement *iii*) follows directly from Theorem 2.6.  $\square$

**Remark 3.6.** We have to remark here that in the statement *iii*) we can not choose the code  $C_{m_b}(H_{m_a}^{q^u})$  (instead of  $C_{m_b}(H_{um_a}^q)$ ), which seems to be natural. We emphasize that the codes  $C_{m_b}(H_{um_a}^q)$  and  $C_{m_b}(H_{m_a}^{q^u})$  are not only different completely regular codes, but they induce different distance-regular graphs with different intersection arrays. So, the code  $C_{m_b}(H_{um_a}^q)$  suits to the codes from *i*) in the sense that it has the same intersection array. For example, the code  $C_2(H_3^{2^2})$  induces a distance-regular graph with intersection array  $(315, 240; 1, 20)$  and the code  $C_2(H_6^2)$  gives a distance-regular graph with intersection array  $(189, 124; 1, 6)$ . To reach these results in both cases we use the same Theorem 2.6.

**Remark 3.7.** The above theorem (Theorem 3.5) can not be extended to the more general case when the alphabets  $\mathbb{F}_{q^a}$  and  $\mathbb{F}_{q^b}$  of component codes  $C_A$  and  $C_B$ , respectively, neither  $\mathbb{F}_{q^a}$  is a subfield of  $\mathbb{F}_{q^b}$  or vice versa  $\mathbb{F}_{q^b}$  is a subfield of  $\mathbb{F}_{q^a}$ . We illustrate it by considering the smallest nontrivial example. Take two Hamming codes, the  $[5, 3, 3]$  code  $C_A$  over  $\mathbb{F}_{2^2}$  with parity check matrix  $H_2^{2^2}$ , and the  $[9, 7, 3]$  code  $C_B$  over  $\mathbb{F}_{2^3}$  with parity check matrix  $H_2^{2^3}$ . Then the resulting  $[45, 41, 3]$  code  $C = C(H_2^{2^2} \otimes H_2^{2^3})$  over  $\mathbb{F}_{2^6}$  is not even uniformly packed in the wide sense, since it has the covering radius  $\rho = 3$  and the outer distance  $s = 7$ , which can be checked by considering the parity check matrix of  $C$ .

#### 4. Completely regular codes with different parameters, but the same intersection array

In [13, Theo. 2.11] it is proved that lifting a  $q$ -ary Hamming code  $C(H_m^q)$  to  $\mathbb{F}_q^s$  we obtain a completely regular code  $C_s(H_m^q)$  which is not necessarily isomorphic to the code  $C_m(H_s^q)$ . However, both codes  $C_s(H_m^q)$  and  $C_m(H_s^q)$  have the same intersection array. As we saw above, the code obtained by the Kronecker product construction, or our extension for the case when the component codes have different alphabets, can have the same intersection array. The next statements are the main results of our paper.

**Theorem 4.1.** *Let  $q$  be any prime number and let  $a, b, u$  be any natural numbers. Then:*

- 1) *There exist the following completely regular codes with different parameters  $[n, k, d; \rho]_{q^r}$ , where  $d = 3$  and  $\rho = \min\{ua, b\}$ :*
  - i)  $C_{ua}(H_b^q)$  over  $\mathbb{F}_q^{ua}$  with  $n = \frac{q^b-1}{q-1}$ ,  $k = n - b$ ;
  - ii)  $C_b(H_{ua}^q)$  over  $\mathbb{F}_q^b$  with  $n = \frac{q^{ua}-1}{q-1}$ ,  $k = n - ua$ ;
  - iii)  $C(H_b^q \otimes H_{ua}^q)$  over  $\mathbb{F}_q$  with  $n = \frac{q^{ua}-1}{q-1} \times \frac{q^b-1}{q-1}$ ,  $k = n - bua$ ;
  - iv)  $C(H_b^q \otimes H_u^{q^a})$  over  $\mathbb{F}_q^a$  with  $n = \frac{q^b-1}{q-1} \times \frac{q^{ua}-1}{q^u-1}$ ,  $k = n - bu$ ;
  - v)  $C(H_b^q \otimes H_a^{q^u})$  over  $\mathbb{F}_q^u$  with  $n = \frac{q^b-1}{q-1} \times \frac{q^{ua}-1}{q^u-1}$ ,  $k = n - ba$ ;
- 2) *All the above codes have the same intersection numbers*

$$b_\ell = \frac{(q^b - q^\ell)(q^{ua} - q^\ell)}{(q - 1)}, \ell = 0, \dots, \rho - 1, \quad c_\ell = q^{\ell-1} \frac{q^\ell - 1}{q - 1}, \ell = 1, \dots, \rho.$$

- 3) *All codes above coming from Kronecker constructions are completely transitive.*

PROOF. The first two codes are obtained by the known lifting construction of the corresponding perfect codes and they all come from Theorem 2.6. The third code is obtained by the known Kronecker product construction (both components have the same alphabet) and come from Theorem 2.4. The two last codes are obtained by the Kronecker construction when the two component codes have different alphabets ( $q$  and,  $q^a$  or  $q^u$ ) and come from Theorem 3.5.

For every code we find the covering radius and compute the intersection array using the corresponding expressions given in the quoted theorems. All these codes have covering radius  $\rho = \min\{ua, b\}$ .

Complete transitivity of all codes coming from Kronecker constructions follows from Theorem 2.4 and Theorem 3.5.  $\square$

It is easy to see that the number of different completely transitive (and, therefore, completely regular) codes with different parameters and the same intersection array is growing. To be more specific we summarize the results in the following Corollary, which comes straightforwardly from the above Theorem 4.1.

For a given natural number  $n$ , let  $n_{(i)}$  be any divisor of  $n$  and  $n^{(i)} = \frac{n}{n_{(i)}}$ . Denote by  $\tau(n)$  the number of divisors of  $n$ .

**Corollary 4.2.** *Given a prime power  $q$  choose any two natural numbers  $a, b > 1$ . We can build the following completely regular codes with identical intersection array and covering radius  $\rho = \min\{a, b\}$ . Specifically, we construct a code over  $\mathbb{F}_{q^r}$  for each divisor  $r$  of  $a$  or  $b$  (the number of different codes is upper bounded by  $\tau(a) + \tau(b)$ ) and we obtain:*

- i) *Completely transitive codes  $C(H_{a_{(i)}}^{q^{a_{(i)}}} \otimes H_b^q)$  over  $\mathbb{F}_{q^{a_{(i)}}}$ , for any divisor  $a_{(i)}$  of  $a$ ,  $a_{(i)} \neq 1$ .*

- ii) Completely transitive codes  $C(H_a^q \otimes H_{b(i)}^{q^{b(i)}})$  over  $\mathbb{F}_{q^{b(i)}}$ , for any divisor  $b(i)$  of  $b$ ,  $b(i) \neq 1$ .
- iii) Completely regular codes  $C_a(H_b^q)$  over  $\mathbb{F}_{q^a}$  and  $C_b(H_a^q)$  over  $\mathbb{F}_{q^b}$ .

## 5. Uniformly packed codes

Recall that a trivial  $q$ -ary repetition  $[n, 1, n]_q$ -code is a perfect code if and only if  $q = 2$  and  $n$  is odd. Denote by  $R_n^q = [I_{n-1} | \mathbf{1}^T]$  the parity check matrix of the  $q$ -ary repetition  $[n, 1, n]_q$  code. The following statement generalizes the corresponding result of [12].

**Theorem 5.1.** *Let  $C(H_m^{q^u})$  be the  $q^u$ -ary (perfect) Hamming  $[n, k, 3]_{q^u}$ -code of length  $n_a = (q^{um} - 1)/(q^u - 1)$  and  $C(R_{n_b}^q)$  be the repetition  $[n_b, 1, n_b]_q$ -code, where  $q$  is a prime power,  $u \geq 1$ ,  $m \geq 2$ ,  $4 \leq n_b \leq (q^u - 1)n_a + 1$ .*

- *The code  $C = C(H_m^{q^u} \otimes R_{n_b}^q)$  is a  $q^u$ -ary uniformly packed (in the wide sense)  $[n, k, d]_{q^u}$ -code with covering radius  $\rho = n_b - 1$  and parameters*

$$n = n_a n_b, \quad k = n - m(n_b - 1), \quad d = 3. \quad (12)$$

- *The code  $C$  is not completely regular.*

PROOF. First, we find the outer distance of code  $C$ . Using [12, Lemma 4] we see that any linear combination of rows of  $H_m^{q^u}$  has weight either  $q^{u(m-1)}$  or zero. By the same arguments used in [12, Theo. 3] (any row of the parity check matrix of  $C(R_{n_b}^q)$  adds one more value to the weight of  $C^\perp$ ) we conclude that  $s(C) = n_b - 1$ .

Now, about the covering radius of code  $C$ , we claim that for the case  $n_b \leq (q^u - 1)n_a + 1$  we have  $\rho = n_b - 1$ .

Since  $\rho(C) \leq s(C)$  (Lemma 2.1), it is enough to show that  $\rho(C) \geq n_b - 1$ . Take an arbitrary column vector  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n_b-1})^T$  where  $\mathbf{x}_i$  is a vector of length  $m$  over  $\mathbb{F}_{q^u}$ . Present this vector as a linear combination of columns of  $H_m^{q^u} \otimes R_{n_b}^q$ . For any vector  $\mathbf{x}_i$  there is a column of  $H_m^{q^u}$  which differs from  $\mathbf{x}_i$  by a scalar. Choose as vectors  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n_b-1}$  all possible different vectors of length  $m$  over  $\mathbb{F}_{q^u}$ . This vector can be presented as a linear combination, at least, of  $n_b - 1$  columns of  $H_m^{q^u} \otimes R_{n_b}^q$ . Hence,  $\rho \geq n_b - 1$  and by Lemma 2.1 we conclude that  $\rho = n_b - 1$ , and, therefore, the resulting code is uniformly packed code.

Since  $(q^u - 1)n_a + 1 = q^{um}$  for the case when  $n_b \geq (q - 1)n_a + 2$ , we can not choose all vectors  $\mathbf{x}_i$  such that they are different. So, if  $n_b = (q - 1)n_a + 2$ , for example, then two subvectors  $\mathbf{x}_i$  and, say,  $\mathbf{x}_j$  should be the same. Now take as columns of  $H_m^{q^u} \otimes R_{n_b}^q$  the column  $\mathbf{h} = (\mathbf{h}_1, \dots, \mathbf{h}_{n_b-1})^T$  with the same subcolumns  $\mathbf{h}_i = \mathbf{h}_j$ . As a result we obtain  $\rho = n_b - 2$ , but  $s = n_b - 1$ , implying that the resulting code is not uniformly packed.

To complete the proof we have to show that  $C$  is not completely regular. This comes by the same argument used in [12].

## 6. Coset distance-regular graphs

Following [3], we give some facts on distance-regular graphs. Let  $\Gamma$  be a finite connected simple graph (i.e., undirected, without loops and multiple edges). Let  $d(\gamma, \delta)$  be the distance between two vertices  $\gamma$  and  $\delta$  (i.e., the number of edges in the minimal path between  $\gamma$  and  $\delta$ ). The *diameter*  $D$  of  $\Gamma$  is its largest distance. Two vertices  $\gamma$  and  $\delta$  from  $\Gamma$  are *neighbors* if  $d(\gamma, \delta) = 1$ . Define

$$\Gamma_i(\gamma) = \{\delta \in \Gamma : d(\gamma, \delta) = i\}.$$

An *automorphism* of a graph  $\Gamma$  is a permutation  $\pi$  of the vertex set of  $\Gamma$  such that, for all  $\gamma, \delta \in \Gamma$  we have  $d(\gamma, \delta) = 1$  if and only if  $d(\pi\gamma, \pi\delta) = 1$ . Let  $\Gamma_i$  be the graph with the same vertices of  $\Gamma$ , where an edge  $(\gamma, \delta)$  is defined when the vertices  $\gamma, \delta$  are at distance  $i$  in  $\Gamma$ . Clearly,  $\Gamma_1 = \Gamma$ . A graph is called *complete* (or a *clique*) if any two of its vertices are adjacent. A connected graph  $\Gamma$  with diameter  $D \geq 3$  is called *antipodal* if the graph  $\Gamma_D$  is a disjoint union of cliques [3].

**Definition 6.1.** [3] *A simple connected graph  $\Gamma$  is called distance-regular if it is regular of valency  $k$ , and if for any two vertices  $\gamma, \delta \in \Gamma$  at distance  $i$  apart, there are precisely  $c_i$  neighbors of  $\delta$  in  $\Gamma_{i-1}(\gamma)$  and  $b_i$  neighbors of  $\delta$  in  $\Gamma_{i+1}(\gamma)$ . Furthermore, this graph is called distance-transitive, if for any pair of vertices  $\gamma, \delta$  at distance  $d(\gamma, \delta)$  there is an automorphism  $\pi$  from  $\text{Aut}(\Gamma)$  which moves this pair  $(\gamma, \delta)$  to any other given pair  $\gamma', \delta'$  of vertices at the same distance  $d(\gamma, \delta) = d(\gamma', \delta')$ .*

The sequence  $(b_0, b_1, \dots, b_{D-1}; c_1, c_2, \dots, c_D)$ , where  $D$  is the diameter of  $\Gamma$ , is called the *intersection array* of  $\Gamma$ . The numbers  $c_i, b_i$ , and  $a_i$ , where  $a_i = k - b_i - c_i$ , are called *intersection numbers*. Clearly  $b_0 = k$ ,  $b_D = c_0 = 0$ ,  $c_1 = 1$ .

Let  $C$  be a linear completely regular code with covering radius  $\rho$  and intersection array  $(b_0, \dots, b_{\rho-1}; c_1, \dots, c_\rho)$ . Let  $\{B\}$  be the set of cosets of  $C$ . Define the graph  $\Gamma_C$ , which is called the *coset graph of  $C$* , taking all different cosets  $B = C + \mathbf{x}$  as vertices, with two vertices  $\gamma = \gamma(B)$  and  $\gamma' = \gamma(B')$  adjacent if and only if the cosets  $B$  and  $B'$  contain neighbor vectors, i.e., there are  $\mathbf{v} \in B$  and  $\mathbf{v}' \in B'$  such that  $d(\mathbf{v}, \mathbf{v}') = 1$ .

**Lemma 6.2.** [3, 11] *Let  $C$  be a linear completely regular code with covering radius  $\rho$  and intersection array  $(b_0, \dots, b_{\rho-1}; c_1, \dots, c_\rho)$  and let  $\Gamma_C$  be the coset graph of  $C$ . Then  $\Gamma_C$  is distance-regular of diameter  $D = \rho$  with the same intersection array. If  $C$  is completely transitive, then  $\Gamma_C$  is distance-transitive.*

From all different completely transitive codes described above in Theorem 4.1, we obtain distance-transitive graphs with classical parameters (see [3]). These graphs have  $q^{ua}$  vertices, diameter  $D = \min\{ua, b\}$ , and intersection array given by

$$b_l = \frac{(q^{ua} - q^l)(q^b - q^l)}{(q - 1)}; \quad c_l = q^{l-1} \frac{q^l - 1}{q - 1},$$

where  $0 \leq l \leq D$ .

Notice that bilinear forms graphs [3, Sec. 9.5] have the same parameters and are distance-transitive too. These graphs are uniquely defined by their parameters (see [3, Sec. 9.5]). Therefore, all graphs coming from the completely regular and completely transitive codes described in Theorem 4.1 are bilinear forms graphs. We did not find in the literature (in particular in [5], where the association schemes, formed by bilinear forms, have been introduced, the description of these graphs, as many different coset graphs of different completely regular codes. It is also known that these graphs are not antipodal and do not have antipodal covers (see [3, Sec. 9.5]). This can also be easily seen from the proof of Lemma 3.1. Indeed, a given vector  $\mathbf{x} \in C(\rho)$  has many neighbors in  $C(\rho)$ .

**Theorem 6.3.** *Let  $C_1, C_2, \dots, C_k$  be a family of linear completely transitive codes constructed by Theorem 3.5 and let  $\Gamma_{C_1}, \Gamma_{C_2}, \dots, \Gamma_{C_k}$  be their corresponding coset graphs. Then:*

- i) *Any graph  $\Gamma_{C_i}$  is a distance-transitive graph, induced by bilinear forms.*
- ii) *If any two codes  $C_i$  and  $C_j$  have the same intersection array, then the graphs  $\Gamma_{C_i}$  and  $\Gamma_{C_j}$  are isomorphic.*
- iii) *If the graph  $\Gamma_{C_i}$  has  $q^m$  vertices, where  $m$  is not a prime, then it can be presented as a coset graph by several different ways, depending on the number of factors of  $m$ .*

PROOF. The proofs are straightforward. Given a completely transitive code  $C_i$ , constructed by Theorem 3.5, we conclude that the corresponding coset graph is distance-transitive with the same intersection array (Lemma 6.2). Then, by using [3, Sec. 9.5], we conclude that this graph is uniquely defined by their parameters and, therefore, it is induced by bilinear forms. Since two codes  $C_i$  and  $C_j$  with the same intersection arrays induce two coset graphs with the same parameters, we conclude that the corresponding graphs  $\Gamma_{C_i}$  and  $\Gamma_{C_j}$  are isomorphic. The last statement follows from Theorem 4.1, since it gives codes with the same intersection array.  $\square$

## 7. Conclusions

In the current paper we use the Kronecker product construction [12] for the case when component codes have different alphabets and connect the resulting completely regular codes with codes obtained by lifting  $q$ -ary perfect codes. This gives several different infinite classes of completely regular codes with different parameters and with identical intersection arrays. Given a prime power  $q$  and any two natural numbers  $a, b$ , we construct completely transitive codes over different fields with covering radius  $\rho = \min\{a, b\}$  and identical intersection array, specifically, one code over  $\mathbb{F}_{q^r}$  for each divisor  $r$  of  $a$  or  $b$ . We prove that the corresponding induced distance-regular coset graphs are equivalent. In other words, the large class of distance-regular graphs, induced by bilinear

forms [5], can be obtained as coset graphs from different non-isomorphic completely regular codes (either obtained by the Kronecker product construction from perfect codes over different alphabets, or obtained by lifting perfect codes [13]). Similar results are obtained for uniformly packed codes in the wide sense. Under the same conditions, explicit construction of an infinite family of  $q$ -ary uniformly packed codes (in the wide sense) with covering radius  $\rho$ , which are not completely regular, is also given.

Finally, an open question arises: *are bilinear forms graphs the only distance-transitive graphs which have such many different presentations as coset graphs?*

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